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Optimal discharging in a branched estuary

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Abstract. For wastewater discharged into one branch of a narrow estuary, the resulting maximum concentration or temperature can vary markedly depending upon the proximity of the discharge site to the branching and upon how the rate of discharge is adjusted. Explicit formulae are derived for the optimal discharge rate to minimize the maximum concentration or temperature experienced in the estuary, while disposing of a given total wasteload over a tidal period. Graphs are used to show the approximately factors of two reductions in that minimized maximum concentration or temperature when the second branch is large, the discharge close to the branching, the decay rate large or the mean river flow large. By optimizing with respect to one pollutant, there is a reasonably wide range of other pollutants for which the environmental impact is nearly minimized.

Key words: estuary environmental impact, pollution, storage effects, mixing

1. Introduction

If wastewater discharges into an estuary cannot be avoided, then the discharge rate should be adjusted to minimize the environmental impact. A traditional method is to ensure that the effluent never returns to the discharge site. Wastewater is accumulated in holding tanks until the early ebb tide when the waste is discharged at a rate proportional to the out-going tide. Webb and Tomlinson [1] drew attention to the prolonged non-return time range and the reduced environmental impact when the discharge site is within a tidal excursion of the open sea.

A more stringent reduction of the environmental impact is to minimize the maximum concentration experienced anywhere in the estuary [2–5]. For narrow estuaries, achieving this ‘minimax’ (minimized maximum) requires the discharge rate from the holding tanks to be adjusted so that as the estuarine water passes the discharge site, the cross-sectionally averaged concentration is brought back up to the minimax value. Elsewhere in the flow, evaporative heat loss or other decay processes will have gradually lowered the concentration from the minimax. Previous investigations of the minimax [2, 3, 4, 5] have been restricted to discharges more than a tidal excursion from any branching or from the open sea.

The purpose of the present paper is to determine how the minimax discharge rate in a narrow estuary is modified when the discharge site is closer than a tidal excursion inland from a branching or from the open sea. As the tide goes out, the mixing between the different bodies of water dilutes the pollutant. So, on the returning tidal flow the water that returns to the discharge site is less polluted than would have been the case in the absence of branching. One illustrative example reveals that the minimax concentration can be as much as a factor of seven lower than the maximum concentration for a non-returning discharging proportional to the out-going flow [1]. Other examples show that the minimax concentration can vary by factors of two depending upon the relative size of the branches or upon the proximity to the branching.

2. Mathematical model

Mathematical modelling for concentration surges in reversing flows can involve intrinsically difficult mathematics [6]. Fortunately, such surges are avoided when there is optimal discharging and simplifications can be justified which lead to explicit results for the optimal discharge rate.

If cross-sectional mixing within each branch of the estuary takes place more rapidly than tidal oscillations or decay (narrow estuaries less than 200 m wide), then concentration variations across the estuary are negligible. The repeatedly-dividing fractal character of classical estuarine systems (such as the Chesapeake Bay) implies that most of the shoreline is in narrow estuaries. Similarly, for any specific contaminant, most of individual reaches will not have any discharge. For the cross-sectionally averaged concentration $c(x, t)$, or temperature above ambient, the usual mathematical model [7–10] is an advection-diffusion equation with decay

$$\partial_t c + \lambda c + u\partial_x c - D\partial_x^2 c = 0. \tag{2.1}$$

Here t is time, x seaward distance, $\lambda(t)$ the decay rate, $u(x, t)$ the bulk velocity, and $D(x, t)$ the longitudinal dispersion coefficient.

The minimax discharging policy [2, 3, 5], keeps the concentration immediately downstream of the discharge location at the constant minimax value. The spatial distribution of concentration is also nearly flat as the tidal flow carries the gradually decaying pollutant away from the discharge site. Away form the discharge, the smallness of the concentration gradient and of the second derivative $\partial_x^2 c$, makes the longitudinal dispersion term $-D\partial_x^2 c$ in equation (2.1) much less important than usual. Giles [3] showed that when the discharging is optimal or near-optimal, the errors are very small if instead of equation (2.1) use is made of the simpler equation

$$\partial_t c + \lambda c + u\partial_x c = 0. \tag{2.2}$$

It deserves comment that for the penetration of pollutant more than a tidal excursion inland of the discharge (or more than a tidal excursion inland of the junction for the branch with no discharge) the diffusive term $\partial_x^2 c$ in equation (2.1) does become important. The present work can be thought of as being an inner representation on the length scale of the tidal excursion and a time scale comparable with the tidal period. The work of O'Connor [10] can be thought of as an outer long scale and tidally averaged represention. Matching the models is not attempted here.

Another simplifying assumption is that the volumetric discharge rate $q(t)$ at the discharge location $x = a$ is small relative to the tidal volume flux of water $A(a, t)u(a, t)$, where $A(x, t)$ is the estuary cross-sectional area. Conveniently, the explicit formulae derived in Section 4 for the optimal discharge rate have $q(t)$ proportional to $A(a, t)u(a, t)$. So, it suffices that the total volume of wastewater to be discharged per tidal period is small relative to the total volume of tidal water passing the discharge site per tidal period (tens of thousands of cubic metres of wastewater to tens of millions of cubic metres of estuary water in an approximately 12 hour tide).

A mass or heat balance, from immediately inland $x = a_-$ to immediately seaward $x = a_+$ of a point discharge at $x = a$, gives a jump in the cross-sectionally averaged concentration:

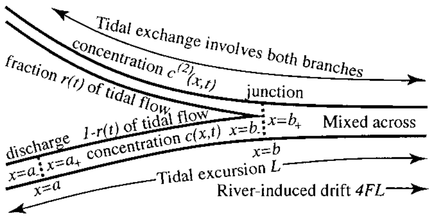

Figure 1. Definition sketch for a discharge site in one branch of a narrow estuary.

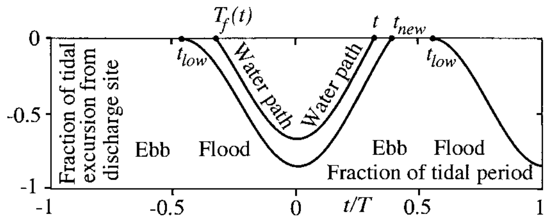

Figure 2. Transition time t_{new} for the arrival of water which has not previously been at discharge. For $0 < t < t_{\mathrm{new}}$ the previous time the water was at the discharge is denoted $T_f(t)$.

$$A(a,t)u(a,t)[c(a+,t) - c(a-,t)] = \gamma(t)q(t), \tag{2.3}$$

where $-\gamma(t)$ is the concentration or temperature of the effluent immediately prior to discharge into the estuary. If the required jump in concentration is known (*i.e.* to maintain the constant minimax value), then Equation (2.3) determines the corresponding discharge rate.

On the incoming flood tide pollutant can penetrate up to one tidal excursion into the branch of the estuary with no discharge. In that second branch, we denote the cross-sectionally averaged concentration by $c^{(2)}(x,t)$. We assume that within the second branch the pollutant continues to decay at the same decay rate $\gamma(t)$ as in the principal branch. So, on the next tide when the pollutant is carried out past the junction, the concentration $c^{(2)}(x,t)$ will have decayed.

To avoid the introduction of more superscripts, the area and velocity seaward of the junction are denoted $A(x,t)$ and $u(x,t)$. Since estuary water is neither created nor destroyed at the junction $x = b$, the sum of volume fluxes just inland of the junction is exactly equal to the volume flux just seaward:

$$A(b_-,t)u(b_-,t) + A^{(2)}(b_-,t)u^{(2)} = A(b_+,t)u(b_+,t). \tag{2.4a}$$

At the junction we shall assume that the flow directions in the two branches are the same. It is only for deep wide and extremely long estuaries that prolonged phase lags between slack water timing in adjacent branches allows substantial direct water and pollutant exhange from one branch to the other without an intervening tidal excursion seaward of the junction [11]. As indicated in Figure 1, from just inland $x = b_-$ to just seawards $x = b_+$ of the junction, the fractions of the combined tidal volume fluxes in each of the two branches are denoted

$$1 - r(t) = \frac{A(b_-,t)u(b_-,t)}{A(b_+,t)u(b_+,t)} \quad \text{and} \quad r(t) = \frac{A^{(2)}(b_-,t)u^{(2)}(b_-,t)}{A(b_+,t)u(b_+,t)}. \tag{2.4b}$$

The assumption about the flow directions implies that $r(t)$ lies between 0 and 1. In the limit as r tends to 0 there is negligible flow in the second branch and in the limit as r tends to 1 there is much larger volume flow than in the branch containing the discharge.

In the numerical examples from Section 5 onwards, r is assumed to be constant. So there is exact phase matching between the tidal fluxes entering or leaving the two branches, and not just matching of the timings for slack water. The narrow estuaries are also assumed to be short relative to a tidal wavelength, so all further inland branches rise and fall in synchrony.

Seaward of where the two branches merge, we assume that mixing across the combined estuary is sufficiently rapid [12] that equation (2.2) can again be used with the same decay rate $\lambda(t)$. On the seaward flowing ebb the concentration just seaward $x = b_+$ of the junction is given by the volumetric mixture of the branch concentrations:

$$c(b_+, t) = [1 - r(t)]c(b_-, t) + r(t)c^{(2)}(b_-, t) \quad \text{on ebb}. \tag{2.5a}$$

The water in the branch with no discharge will tend to be of lower concentration than the water in the branch where the discharge is made. Thus, mixing at the junction tends to cause a drop in the cross-sectionally averaged concentration as the lower concentration $c^{(2)}$ water dilutes the higher concentration c water.

On the inland flowing flood the cross-sectionally well-mixed water just seawards $x = b_+$ of the junction is shared in the volumetric ratio $1 - r(t) : r(t)$ between the two branches and arrives just inland $x = b_-$ with unchanged concentration:

$$c(b_-, t) = c(b_+, t) \quad \text{and} \quad c^{(2)}(b_-, t) = c(b_+, t) \quad \text{on flood}. \tag{2.5b}$$

3. Pollution-history representations

To solve Equations (2.2–2.5) we shall investigate the pollution history of the water (*i.e.* previous times at the discharge and mixing events at the junction.

Figure 2 identifies the time t_{new} on late ebb when the water arriving at the discharge site ceases to have returned from a previous departure on the previous flood. Prior to t_{new} the water arriving at time t from just inland of the discharge (a_-, t) will have previously departed at time $T_f(t)$ from the discharge during flood $(a_-, T_f(t))$. In the intervening time, the concentration will have decayed:

$$c(a_-, t) = c(a_-, T_f(t))E(T_f(t), t) \quad \text{on early ebb} \quad 0 < t < t_{\text{new}} \tag{3.1a}$$

The decay factor $E(T_{\text{start}}, t)$ between times T_{start} and t involves the integral of the decay rate over the intervening time $T_{\text{start}} < t' < t$:

$$E(T_{\text{start}}, t) = \exp\left(-\int_{T_{\text{start}}}^{t} \lambda(t')\,\mathrm{d}t'\right). \tag{3.2}$$

Between t_{new} and low water slack t_{low} the water moving seaward has never previously been at the discharge. For the non-diffusive model, this new water has the zero concentration that it had when it first came from a river into the estuary

$$c(a_-, t) = 0 \quad \text{on late ebb (new water)} \quad t_{\text{new}} < t < t_{\text{low}}. \tag{3.1b}$$

Figure 3 identifies transition times t_0, t_1 when the number of previous junction mixing events changes. Figure 3 also illustrates that the water which is returning to the discharge site on ebb tide at a time t slightly later than the transition time t_1, has participated junction mixing events at the two times $\tau_1(t), \tau_2(t)$ and the pollutant can be traced to the two previous times of discharge $T_1(t), T_2(t)$. To avoid superposition of curves in the lower part of Figure 3, the excursion distance in the second branch (dotted curves) is shown as being greater than in the principal branch. The indicated distances are from the junction, by contrast to Figure 2 where the indicated distances are from the discharge location .

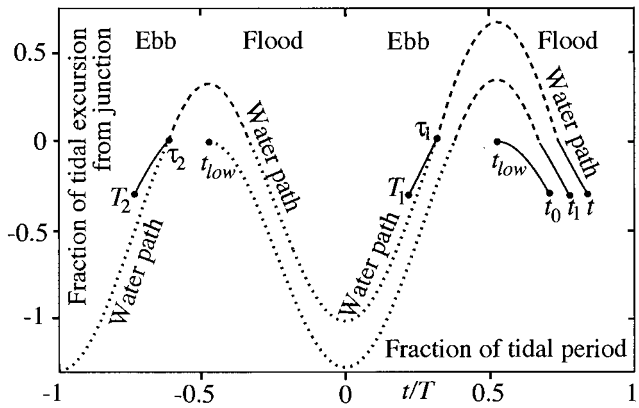

Figure 3. Transition times t_0 and t_1 for increasing numbers of junction mixing events τ_1, τ_2 and of contributing times of discharge T_1, T_2. The continuous, dotted and dashed curves respectively follow the water in the principal, second and combined branches of the estuary.

Sufficiently early on flood $t_{\text{low}} < t < t_0$, the water moving inland from just seaward of the discharge (a_+, t) will not have experienced a mixing event at the junction $x = b$ since it departed from the discharge on the previous ebb $(a_+, T_0(t))$. The concentration will have decayed:

$$c(a_+, t) = c(a_+, T_0(t))E(T_0(t), t) \quad \text{on early flood} \quad t_{\text{low}} < t < t_0, \tag{3.1c}$$

The subscript $_0$ in $T_0(t)$ and in t_0 is an indicator that there have been zero mixing events at the junction for the appropriate water masses.

Later on flood, the returning water at the discharge will have participated in some number $N \geq 1$ of ebb-tide junction mixing events. We shall denote the times of those junction mixing events $\tau_N(t), \ldots, \tau_1(t)$, as illustrated in Figure 3. It is water from the most recent junction mixing event $c(b_+, \tau_1(t))$ that is returning to the discharge partially decayed.

$$c(a_+, t) = c(b_+, \tau_1(t))E(\tau_1(t), t). \tag{3.1c}$$

The volumetric mixing (2.5a) relates $c(b_+, \tau_1(t))$ to a combination of $c(b_-, \tau_1(t))$ and $c^{(2)}(b_+, \tau_1(t))$. Within the $^{(2)}$ branch we can use the pollution history to relate $c^{(2)}(b_-, \tau_1(t))$ to the concentration previous mixing event $c(b_+, \tau_2(t))$ if any, or to zero. Within the main branch we can use the pollution history to relate $c(b_-, \tau_1(t))$ to the time $T_1(t)$ earlier on the same ebb when the water had just left the discharge with concentration $c(a_+, T_1(t))$. So, our expression for the concentration returning from the junction can be modified:

$$c(a_+, t) = c(b_+, \tau_2(t))E(\tau_2(t), t)r(\tau_1(t)) + c(a_+, T_1(t))E(T_1(t), t)[1 - r(\tau_1(t))].$$

Repeating the pattern of calculations to replace $c(b_+, \tau_2(t))$, the final composite expression for the concentration returning from the junction involves the concentrations at all N previous discharge times $T_N(t), \ldots, T_1(t)$ and the dilution ratios at the corresponding junction mixing events $\tau_N(t), \ldots, \tau_1(t)$:

$$\begin{aligned} c(a - +, t) = &\sum_{m=2}^{N} c(a_+, T_m(t))E(T_m(t), t)[1 - r(\tau_m(t)] \prod_{n=1}^{m-1} r(\tau_n(t)) + \\ &+ c(a_+, T_1(t))E(T_1(t), t)[1 - r(\tau_1(t)] \quad \text{on late flood} \quad t_{N-1} < t < t_N. \end{aligned} \tag{3.1d}$$

4. Optimum rate of discharging

Having obtained the pollution history representations (3.1a–d) for the concentration just before it arrives back at the discharge location, we can evaluate the jump in concentration to keep the leaving concentration constant (denoted C). The explicit results for the discharge rate then follows from the use of Equation (2.3):

$$\gamma(t)q(t) = CA(t)u(t)\{1 - E(T_f(t), t)\} \quad \text{on early ebb} \quad 0 < t < t_{\text{new}}, \tag{4.1a}$$

$$\gamma(t)q(t) = C(t)Au(t) \quad \text{on late ebb (new water)} \quad t_{\text{new}} < t < t_{\text{low}}, \tag{4.1b}$$

$$\gamma(t)q(t) = CA(t)u(t)\{1 - E(T_0(t), t)\} \quad \text{on early flood} \quad t_{\text{low}} < t < t_0, \tag{4.1c}$$

$$\begin{aligned}\gamma(t)q(t) = CA(t)|u| \Bigg\{ & 1 - E(T_1(t), t) + \sum_{m=1}^{N-1}[E(T_m(t), t) - E(T_{m+1}(t), t)] \\ & \prod_{n=1}^{m-1} r(\tau_n(t)) + E(T_N(t), t)\prod_{m=1}^{N} r(\tau_m(t))\Bigg\} \quad \text{on late flood} \quad t_{N-1} < t < t_N.\end{aligned} \tag{4.1d}$$

For neatness the truncated notations $A(t)$ and $u(t)$ are used to denote the cross-sectonal area $A(a, t)$ and flow velocity $u(a, t)$ at the discharge location $x = a$.

In practice, the tidally averaged discharge rate $\langle q \rangle$ is specified and any time-dependence pre-discharged effluent quality $\gamma(t)$ can be regarded as being known. Thus, the minimax concentration C can be determined from Equations (4.1a–d) by averaging $q(t)$ over a tidal cycle.

Because the decay factor $E(T_{\text{start}}, t)$ decreases as the starting time T_{start} decreases, the coefficients multiplying the r-products Equation (4.1d) are all positive. Thus, mixing at the junction is necessarily associated with increased discharge rate $q(t)$ on late flood. Equivalently, to discharge a given tidally averaged load $\langle q \rangle$, mixing and dilution at the junction reduces the minimax concentration C.

We can gain insight into the nature of the optimal discharging by considering some limiting cases. If the decay in a tidal period is small, Equations (4.1a–d) can be approximated:

$$\gamma(t)q(t) = CA(t)u(t)\int_{T_f(t)}^{t} \lambda(t')\,\mathrm{d}t' \quad \text{on early ebb} \quad 0 < t < t_{\text{new}}, \tag{4.2a}$$

$$\gamma(t)q(t) = CA(t)u(t) \quad \text{on late ebb (new water)} \quad t_{\text{new}} < t < t_{\text{low}}, \tag{4.2b}$$

$$\gamma(t)(t)q = CA(t)u(t)\int_{T_0(t)}^{t} \lambda(t')\,\mathrm{d}t' \quad \text{on early flood} \quad t_{\text{low}} < t < t_0, \tag{4.2c}$$

$$\begin{aligned}\gamma(t)q(t) = \; CA(t)|u(t)| \Bigg\{ & \int_{T_1(t)}^{t} \lambda(t')\,\mathrm{d}t' + \sum_{m=1}^{N-1}\int_{T_m+1(t)}^{t} \lambda(t')\,\mathrm{d}t' \prod_{n=1}^{m-1} r(\tau_n(t)) + \\ & + [1 - \int_{T_n(t)}^{t} \lambda(t')\,\mathrm{d}t'] \prod_{m=1}^{N} r(\tau_m(t))\Bigg\} \quad \text{on late flood.}\end{aligned} \tag{4.2d}$$

At both waters the discharge is zero and increases slowly (4.2a,c) as the returning water has had more time for the decaying process to evolve. The mixing at the junction gives a relatively high discharge rate in late flood (4.2d), which is the counterpart to the new water high discharge rate in late ebb. Towards high slack water, every time the number $N(t)$ of previous ebb tides that contribute to the concentration jumps, there are drops in the optimal discharge rate (4.2d). The surges and drops in discharge rate to achieve flat minimax concentrations are an inversion of the surges and drops in concentration when there is a flat discharge rate [6–9].

When the second branch of the estuary system is the open sea or is much larger than the branch with the discharge, the r-values tend to 1 and the formula (4.2d) simplifies to

$$\gamma(t)q(t) = CA(t)u(t) \quad \text{on late flood} \quad t_0 < t < T. \tag{4.3}$$

Thus, the mixed water returning from the junction is diluted so much that it can be maximally discharged into, exactly the same as with the new water on late ebb (4.1b).

In the absence of river flow there is periodic return to the junction at the same tidal phase infinitely often:

$$\tau_m(t) = \tau_1(t) + (m-1)T, \quad r(\tau_m(t)) = r(\tau_1(t)), \tag{4.4a, b}$$

where T is the tidal period. If the decay rate is constant, or is tidally periodic with average value $\langle\lambda\rangle$ (*e.g.* removal of pollutant by biochemical reaction with the sediments cyclically stirred up by the tidal flow), then the there is a neat explicit expression for the infinite series (4.1d):

$$\gamma(t)q(t) = CA(t)|u(t)| \left\{ 1 - \frac{E(T_1(t), t)[1 - r(\tau_1(t))}{1 - r(\tau_1(t)) \exp(-\langle\lambda\rangle T)} \right\} \quad \text{on late flood} \quad t_0 < t < T. \tag{4.5}$$

This expression is similar to the early ebb result (4.1a). The $\langle\lambda\rangle T$ term in the denominator makes the optimal discharge greater on late flood than on ebb, *i.e.* that fraction of the water that had been in the second branch had its pollution level decaying for a tidal cycle without passing the discharge site and getting the concentration brought back up to the minimax concentration level.

5. Reference example

For the illustrative examples, we take the decay rate λ, cross-sectional area A, effluent concentration γ and ratio r of tidal volume fluxes to be constants. The tidal current is constructed from two sinusoids smoothly matched at both slack waters [5]:

$$u = U(1+F)\sin\left(\frac{2\pi t}{(1+F)T}\right) \quad \text{on ebb} \quad 0 \le \frac{t}{T} \le \frac{1}{2}(1+F), \tag{5.1a}$$

$$u = U(1-F)\sin\left(\frac{2\pi(t-T)}{(1-F)T}\right) \quad \text{on flood} \quad \frac{1}{2}(1+F) \le \frac{t}{T} \le 1. \tag{5.1b}$$

Here U is the amplitude for the tidal velocity, T is the tidal period and F is a dimensionless characterization of the mean flow. When averaged over a tidal period there is a flow $4FU/\pi$ which we can associate with rivers. The velocity in the second branch is given by the formulae (5.1a,b) with different amplitude $U^{(2)}$ for the tidal velocity and different cross-sectional area

$A^{(2)}$ but the same F. For the combined channel seaward of the junction, F remains fixed but the velocity amplitude and cross-sectional again differ.

A natural excursion length scale to associate with the tidal current is the $F = 0$ small river-flow limit of the ebb-tide or flood tide excursion distances

$$L = UT/\pi. \tag{5.2}$$

On the flood the water can have returned from the junction only if

$$\frac{b-a}{L} \le (1-F)^2. \tag{5.3}$$

As a physical example, we consider an estuary with the second branch contributing two-thirds of the combined tidal volume flux. In the first branch the velocity amplitude for the semi-diurnal tide is $U = 0{\cdot}7$ m s^{-1}, giving a natural excursion length $L = 6{\cdot}7$ km. The discharge is $b - a = 2$ km inland of the junction. The most important pollutant has an e-folding decay time of $1/\lambda = 2$ days and the tidally averaged mean velocity towards the sea is $0{\cdot}045$ m s^{-1}. The dimensionless characterization of this physical example is:

$$r = \frac{2}{3}, \quad \frac{b-a}{L} = 0{\cdot}3, \quad \lambda T = 0{\cdot}25, \quad F = 0{\cdot}05. \tag{5.4}$$

This specification (5.4) is used as a reference example in every figure (continuous curves).

6. Previous times

For the illustrative flow (5.1a,b) the arrival time for the new water which has not previously been at the discharge site has an explicit formula:

$$t_{\text{new}} = \frac{(1+F)}{\pi} \arcsin\left\{\frac{1-F}{1+F}\right\}. \tag{6.1a}$$

On early ebb prior to t_{new}, the water returning to the discharge had previously departed from the discharge on flood at the time

$$\frac{T_f(t)}{T} = -\frac{(1-F)}{\pi} \arcsin\left\{\frac{1+F}{1-F} \sin\left(\frac{\pi t}{(1+F)T}\right)\right\} \quad \text{for} \quad 0 \le t \le t_{\text{new}}. \tag{6.1b}$$

For later use, we remark that for small t both the flow velocity $u(t) \approx 2\pi Ut/T$ and the previous time $T_f(t) \approx -t$ become independent of the mean flow parameter F.

On early flood the returning water did not reach the junction and had previously departed from the discharge during ebb at the time

$$\frac{T_0(t)}{T} = \frac{(1+F)}{\pi} \arccos\left\{\frac{1-F}{1+F} \cos\left(\frac{\pi(T-t)}{(1-F)T}\right)\right\} \quad \text{for} \quad \frac{1}{2}(1+F) \le t \le t_0, \tag{6.2a}$$

For $N \ge 1$ junction visits, the previous (ebb) times at the discharge are:

$$\frac{T_m(t)}{T} = 1 - m + \frac{(1+F)}{\pi} \arccos\left\{\frac{1}{1+F}\left[(1-F)^2 \cos^2\left(\frac{\pi(T-t)}{(1-F)T}\right) - 4(m-1)F\right]^{1/2}\right\} \quad \text{for} \quad t_{m-1} \le t \le t_m, m \le N \tag{6.2b}$$

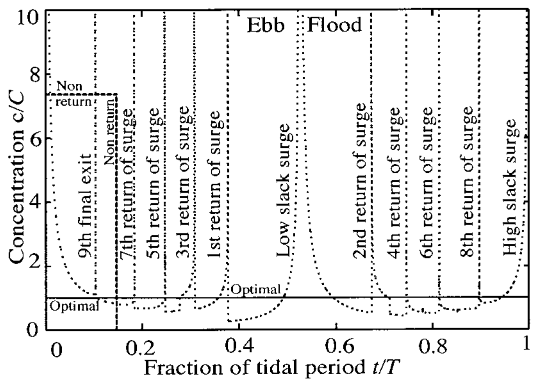

Figure 4. Concentrations at the discharge as given by the zero dispersion model, when there is a steady (dotted), non-returning (dashed) and an optimal discharge (continuous). Dispersion would smooth the returning concentration surges but not the slack-water concentration surges.

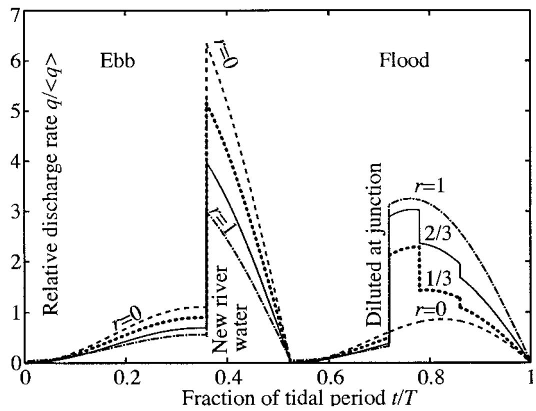

Figure 5. Optimal discharge rates for second branches with different fractions r of the combined tidal flux. The constant concentration maximums are in the same ratios as the discharge rates during ebb and early flood.

The assumed proportionality between the flows in the c-branch and seaward of the junction, results in T_0 and T_1 being identical except in their time domains of application. The transition times for m junction visits are:

$$\frac{t_m}{T} = 1 - \frac{(1-F)}{\pi} \arccos\left\{ \frac{1}{1-F} \left[\frac{b-a}{L} + 4mF \right]^{1/2} \right\}. \tag{6.2c}$$

For the reference example (5.4) the modest river flow and discharge distance from the branching, allows up to $N = 3$ junction visits.

7. Minimax versus non-returning

First we check for the reference case (5.4) whether the optimization is worthwhile. Figure 4 plots the concentration relative to the minimax concentration immediately downstream of the discharge, computed from equations (2.3, 3.1a–d) for three different ways of disposing of the same total amount $\langle q\rangle T$ of wastewater in a tidal period. The dotted curves show the concentrations for a steady discharge at the rate $\langle q\rangle$. At low slack water discharging at a steady rate into stationary water results in a concentration surge. For the reference case $F = 0{\cdot}05$ of moderate river flow, that surge returns a total of nine times. The time duration of the surge at the discharge is less when the flow is faster. The dashed curve shows the concentration relative to the minimax for a non-returning discharge proportional to the flow rate on early ebb [1]. The non-returning strategy eliminates the low-slack and returning surges but leaves an extended flattened high water surge with concentration $7{\cdot}4C$. The continuous line along unity is achieved with the optimal discharge.

The simplification (2.2) of ignoring dispersion becomes inaccurate in the short sharp concentration surges. As the water repeatedly returns up nine times, the spikes should become more and more smeared out [3,4,13]. Bikangaga and Nassehi [4] show that far from the discharge, the severity of the surges diminishes typically to three times the minimax. The inclusion of the longitudinal dispersion term does not remove the most prolonged concentration peaks at low and high slack waters [3, 13]. The only way to do that is by reducing the discharge rate to zero as the tide turns [2–5].

8. Branch sizes

For the illustrative flow, with r and λ constant and with explicit expressions for the previous times $T_f(t)$ and $T_m(t)$, the relationship (4.1 a–d) between the optimal discharge rate $q(t)$ and the minimax concentration C is easy to evaluate.

With the flow model (5.1a,b) four parameters are needed to specify the branched geometry, the discharge location, the pollutant decay and the river flow. Correspondingly, four comparisons are made relative to the reference case (5.4). The parameter ranges are chosen to span the full range in which the presence of a junction influences the discharging.

Figure 5 shows how different r-values

$$r = 0, \frac{1}{3}, \frac{2}{3}, 1. \tag{8.1}$$

change the optimal discharge rates for disposing of a given amount of wastewater $\langle q \rangle T$ per tide. The tidal volume fluxes in the discharge-free branch are respectively negligible, half, twice and vastly greater than that in the channel which contains the discharge. The remaining parameters (location, pollutant and mean flow) are as given in equations (5.4). It is a characteristic feature of optimal discharging [2,3,5] that when unpolluted river water first arrives at the discharge it is greeted by a sudden increase in the rate of discharge.

The number and timing of visits by water masses to the junction and to the source are the same for al four curves in Figure 5. So, the jumps in discharge rates are aligned vertically. It is on in the late flood that the differing amounts of dilution at the junction has a direct influence at the discharge. The relative amount of wastewater discharged in late flood does effect the evaluation for the minimax concentration C and thereby has an indirect effect on the optimal discharge rate at other times.

The size parameter r does not occur in equations (4.1a–c) for the optimal discharge rate $q(t)$. Hence, throughout early ebb, late ebb and early flood the constant ratio between the optimum discharge rates is the same as the ratio between the minimax concentrations C for the four r-values. Thus, from Figure 5 we can see that there is a factor of 2·15 difference in the minimax concentration depending on the size of the second branch.

9. Distance between discharge and junctions

Figure 6 shows how different junction distances

$$\frac{b-a}{L} = 0{\cdot}9, 0{\cdot}6, 0{\cdot}3, 0, \tag{9.1}$$

(6 km, 4 km, 2 km, 0 km) change the optimal discharge rates for disposing of a given amount of wastewater $\langle q \rangle T$ per tide. The parameters r, λT and F for the relative sizes of the branches, the decay per tidal cycle and the mean river flow are as specified in Equations (5.4).

It is on flood tide in the different returning times that the different proximities to the junction have a direct effect back at the discharge. The longer the time that there is diluted

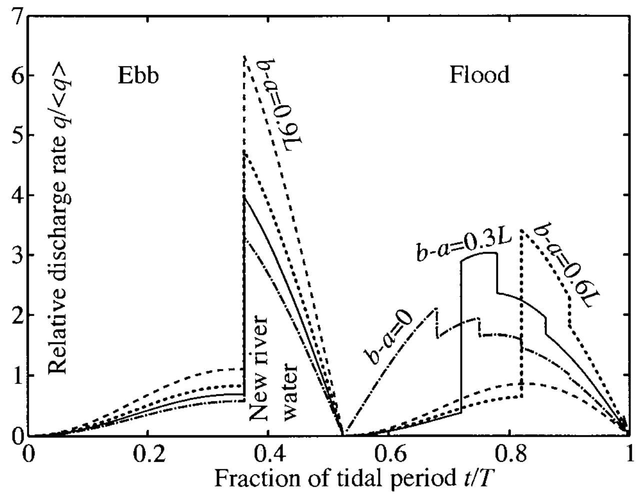

Figure 6. Optimal discharge rates for junctions at different fractional tidal excursions seaward. The constant concentration maximums are in the same ratios as the discharge rates during ebb.

water from the junction, the greater the fraction of the total wasteload can be discharged into that water and the lower the minimax concentration C.

On ebb the previous time (6.1a) is not affected by the presence of any seaward junction. Thus, throughout ebb (4.1a,b) and the early flood (4.1c) (absent for $a = b$) the relative magnitudes of the discharge rates is the same as for the minimax concentrations. Hence, from Figure 6 the factor of 1·9 disparity in discharge rates throughout ebb allows us to infer that there is a factor of 1·9 disparity in the minimax concentration between the extremes of discharges close to and far from the junction.

The saw-tooth shape in flood of the (dot-dash) zero-distance optimal discharge rate can be attributed to the relatively many return times for water with pollutant which has traversed and decayed in the larger discharge-free branch. For a more distant junction the number of return times (and the number of saw-teeth) is reduced.

The reference case (5.4), indicated by the continuous curve, is common to Figures 5–8. However, the dashed curves in Figures 5 and 6 are also the same as each other: a second branch of zero tidal volume flux is just as ineffective at diluting the concentrations as is a discharge site too far inland of the branching for any water to return.

10. Decay rates

Figure 7 shows how different decay rates

$$\lambda T = \infty, 1, 0{\cdot}25, 0, \qquad (10.1)$$

(e-folding decay times zero, half a day, 2 days, non-decaying) change the optimal discharge rates for disposing of a given amount of wastewater $\langle q \rangle T$ per tide. The the parameters r for the size ratio of the two branches, $(b - a)/L$ for the distance from the junction and F for the mean river flow, are as specified in Equation (5.4). It deserves note that for the non-decaying pollutant there is zero discharge throughout the early ebb and the early flood.

As in Figure 5, the number and timing of visits by water masses to the junction and to the discharge are the same for all four cases, making any jumps be aligned vertically. It is only

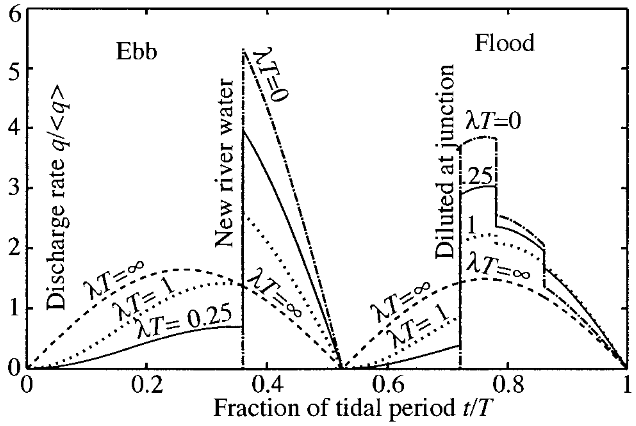

Figure 7. Optimal discharge rates for pollutants with different decay rates per tide. The constant concentration maximums are in the same ratios as the discharge rates in the new water.

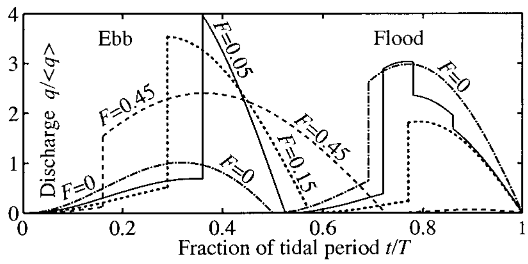

Figure 8. Optimal discharge rates for different mean flows. The constant concentration maximums are in the same ratios as the discharge rates in early ebb.

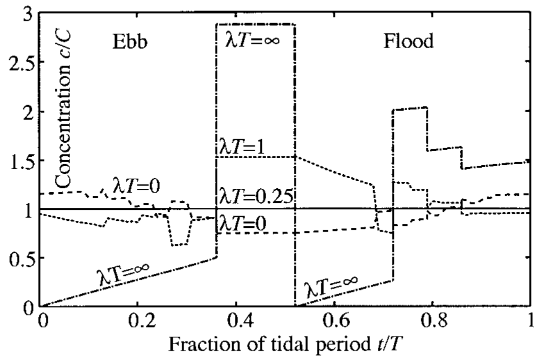

Figure 9. If the discharge rate is optimised for a pollutant with $\lambda T = 0{\cdot}25$, then for pollutants with different λT the concentrations relative to the different minimax concentrations C must exceed 1 for part of the tidal cycle.

in the new water period (4.1b) that the optimal discharge rates are free from any exponential decay factors and independent of the decay rate. Thus, in the new water period the discharge rates in Figure 7 are in the same proportion as the minimax concentrations C. There is a 3·9 range in the minimax concentration between instantly decaying and non-decaying pollutants. The longer lasting the pollutant the more significant the extra dilution provided by the second branch.

11. River flows

Figure 8 shows the effect of varied river flow

$$F = 0{\cdot}45, 0{\cdot}15, 0{\cdot}05, 0, \tag{11.1}$$

(mean flow speeds 0·4m s^{-1}, 0·13 m s^{-1}, 0·045 m s^{-1} no flow). The size ratio for the two branches, the distance from the junction and the decay rate are as specified in equation (5.4).

For zero river flow it is necessary to use the formula (4.5) to accommodate the unlimited number of return times. The most marked change from the previous Figures 5, 6, 7 is that the different river flows result in different new water arrival times in ebb. There are also changes in timing for low slack water and for the return on flood tide of mixed water from the junction.

To assess the relative minimax concentrations we make use of the property noted in Section 6, that for small t both the flow velocity $u(t)$ and the previous time $T_f(t)$ become independent of the mean flow parameter F. Hence we can compare the relative C values by comparing the relative discharge rates (4.1a) as they increase from zero in the early ebb. Zero river flow gives 3·8 times the minimax concentration of the high flow $F = 0{\cdot}45$ extreme.

12. Holding tanks

If the effluent is produced at a steady rate but is discharged at a varying rate, then holding tanks would be needed. The tanks would be being filled while $\langle q \rangle$ is less than unity and emptied while $q/\langle q \rangle$ exceeds unity. The areas between $/\langle q \rangle$ and unity in Figures 5–8 allow a visual or numerical assessment of the volumes for the necessary holding tanks.

In the absence of a junction ($r = O$) the emptying of the holding tanks tends to be restricted to the new water on late ebb (the high parts dashed curves in Figures 5 and 6). Thus, there is storage accumulating throughout the flood and continuing into the early ebb. The necessary holding tank volume is $0{\cdot}43\langle q \rangle T$. The presence of a junction allows some (or complete) emptying on the flood. For the reference case (the continuous curve repeated in Figures 3–8) the necessary holding tank volume is reduced to $0{\cdot}25\langle q \rangle T$. So, not only does the presence of a second branch of an estuary reduce the minimax concentration, but also the engineering task and expense of providing large enough holding tanks is made easier.

13. Several pollutants

Waste water usually contains a variety of pollutants (brine, heat, oxygen demand etc.) with decay rates varying from 0 to ∞. If the discharge rate has been optimized with respect to the $\lambda T = 0{\cdot}25$ species, how far from optimum are the concentration peaks for other pollutant species? Figure 9 plots the maximum concentration (*i.e.* as the water leaves the discharge) for species with the four decay rates (14) relative to the minimax appropriate for that species. The parameters r, $(b - a)/L$ and F are as specified in Equation (5.4). Of course, the relative concentration for the reference species is a horizontal straight line at unity.

For the rapidly decaying pollutant (dash-dot curve) the optimum discharge rate, shown as the dashed curve in Figure 7, would be proportional to the flow speed. Instead, for the results shown in Figure 9, the discharge rate is that appropriate to $\lambda T = 0{\cdot}25$ (shown as the continuous curve in Figures 5, 6, 7, 8) and has a sudden increase in discharge rate when the new water arrives. The consequence for the rapidly decaying pollutant is a sudden increase in concentration to nearly three times the minimax. During flood, there there are further jumps in discharge rate when junction mixed water returns, and corresponding jumps in the concentration for the rapidly decaying pollutant.

For the pollutants with little or no decay (dashed curve) and with decay time one tidal period (dotted curve), the relative departures from the minimax are less severe. The jaggedness corresponds to multiple returning at the nine times labelled in Figure 3. The optimal discharge rate adjustments to the returning for $\lambda T = 0{\cdot}25$ are not optimal for $\lambda T = 1$ or $\lambda t = 0$.

By comparison with the factor of 7·4 increase in the maximum concentration associated with the non-returning policy [1], the pollution events shown in Figure 9 are quite modest. By optimizing with respect to one pollutant, there is a reasonably wide range of other pollutants for which the environmental impact is nearly minimized.

14. Concluding remarks

The accumulative message of this paper together with its antecedents [2,3,4,5] is that an easy, effective and robust way of reducing the environmental impact of unavoidable wastewater discharges in estuaries is to control the discharge rate to match the time- dependent dilution capacity. The particular message of the present paper is that in branched estuaries a large contribution to the dilution capacity is the mixing on ebb flow between the tidal waters from the different branches.

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